1

Chapter of Logic

The study of geometry is important because it can provide a way for students to develop logical thinking. The skills of logical thinking go far beyond mathematics and are tools that students will have for life. While this approach is an important foundation for the work in geometry, we can start to cultivate that type of thinking through games and puzzles that are unrelated to geometry. It is important throughout all these games and activities that students articulate what they are thinking and why. This is because we want them to be aware of the *process,* which for these games is more important than the answer or outcome. Doing work in groups also allows students to hear what others are thinking and to have discussions that strengthen their own though processes. They can begin developing ways of thinking before they have specific skill knowledge. These activities can be done at a young age, with various levels of sophistication brought in as students are ready.

When students work in Montessori elementary, they often use an inductive reasoning approach. For example, students will perform a math experiment several times, will notice patterns, and will draw conclusions about them to come up with a rule. Eventually, students will switch to deductive reasoning, which is more formal and no longer based on examples, but on pure logic that can be followed through from first principles. While some deductive thinking may be done early on, through observing adolescents of all ages, it seems that there is more of an interest and a need for this type of thinking around age 15. Until then, a strong inductive approach is important and satisfactory. Often trying to take a deductive approach to an idea seems repetitive, irrelevant, and boring. For example, let's say students want to discover (or rediscover) the idea that the sum of the interior angles of a triangle measures 180°. An activity where students measure all three angles of various triangles and sum them will show that they always total (close to) 180°. They can also then tear off the three corners and arrange them into a straight line, thus supporting their hypothesis. This conclusion is correct and is a powerful visual to remind them of what the measure of the interior angles actually is. If they forget the measure, they can recall easily their experiment and be confident the answer will be 180°. However, this is just an experiment, and we have not proven beyond a reasonable doubt that the sum will always be 180°. To do this, we would need to do a formal proof. For the younger adolescent, who already knows and believes the answer, there does not seem to be much gained by this exercise. They can follow along, but at the end, the response may be, "But I already knew that! What more was there to gain by doing this exercise?" They may dismiss the deductive proof as tedious and unnecessary because it did not give them more knowledge than when they started. They already knew the answer, and they can even show that it always works! Why do they need to do something that is more complicated for no gain?

As students mature through adolescence, they start to appreciate and even desire more rigor in their work. The idea that a deductive proof gives a different level of knowledge and understanding than before is now appealing to them. They are now privy to a fundamental truth, which cannot be disputed or taken away. There is something beautiful and powerful about knowing an absolute truth. Even though they have no more new practical geometric knowledge, it is about having a new level of understanding about the universe. To no longer *believe* something is true, but to *know* that it is true is something that is now appreciated and can be a motivator for the student to build this logical thinking going forward.

To help students build their deductive reasoning skills, we start early by giving them activities and questions to help build these skills. We also try to appeal to their imaginations and to their sense of philosophy, which is extremely attractive to them at this age. We talk about the Greeks and their search for truth. Many educated Greeks were philosophers, and if we look at the Timeline of Mathematics, many of the Greeks pictured there had a variety of occupations; mathematician and philosopher being two of them. This is no coincidence, as mathematics actually came out of philosophy. The

Greeks would ask themselves question like, “What is truth?” This can only be answered by thinking. But how would they know if they were right? They tried to confirm what they thought by reconciling that with the world around them. A philosophical question like, “What is the smallest object?” could be thought of as asking, “If we took this piece of bread and divided it in half again and again and again, and we could have as sharp a knife as we needed, what would happen? How small could we get? Is there a smallest thing?” For this question, they concluded that eventually you would get to the smallest, indivisible particle: the *atom,* from *a-* meaning *not*, and *tomos*, meaning *a cutting*. Even their words tell us how they thought about this question. However, the important thing here is that their thinking led them to try to understand the nature of the universe. They developed tools to help them see if their thinking was correct and this is how mathematics as we see it today was developed.

The Greeks believed there was an absolute truth to the nature of the universe and they wanted to uncover it. However, this truth was only available to the mind. The world around them was only a reflection of the principles they sought to prove, not the proof itself. Absolute truth came through the abstraction of thought, while the world itself was only relative truth. I will ask my students, “How old are you?” When they say “fourteen,” I say, “Happy birthday!” When they say it is not their birthday, I will call them a liar because they said they were fourteen years old. When they try to be more precise, down to the month or the day, like, “Fourteen years, two months, and four days,” I will say something like, “So you were born at 10:34 a.m.?” The point of this questioning is that there is an exact age for every person, but knowing it with precision isn’t easy. The absolute truth of their age exists, but we may never have access to that information. However, how important is this? Can’t we live our lives with less than perfect knowledge? Yes, we can, and it is often preferable on a practical level. The student wasn’t lying when she said she was fourteen years old; she gave me a truth in a practical way that is meaningful for what is important to us in our society. However, we are drawn to the idea of a universal truth. How can we access this? We can *imagine* that there is a precise moment of birth and that we could calculate the lifespan of a person *if* we had that knowledge. We know only in our minds that this ultimate truth exists, even though we can never access it in actuality.

The Greeks tried to determine what their reality really was. For example, they most certainly saw circles everywhere and asked, "What is a circle?" They might define it as a collection of points all equidistant from another point, which we call the center. In our minds, and by our definition, we understand what a circle is. It makes logical sense and definitely describes our *abstraction* of the idea of a circle from all our observations. But no one has ever seen a true circle. If we had very powerful measuring tools, we would see that the circle in front of us is not symmetrical. Some distances from what we think is the center will be different from others. If we use a microscope to examine the circumference of our circle, we would see the edge is not smooth, but ragged and uneven. Indeed, if we got down to the atomic level, there would not even be an edge at all, just random spots of energy with tremendous gaps of nothingness in between. Yet we can imagine a perfect circle, we can see it in our mind's eye, and it describes very well our idea of every physical circle that we see in our universe. We have abstracted the essence of nature, and through our thinking have uncovered the ideal circle. What exists in our mind is the ultimate truth, but it only exists there. Although the perfect circle exists only in our minds, it describes our world to any practical degree that we wish, just like the question about one's age.

Having students reflect on these ideas invites them in to something beyond what we traditionally think of as mathematics, but which is really the field's origins! It is thinking about the truth of our world, and how we can describe it in ways that are general and precise using abstractions from what our world actually is. Besides circles, other things to think about are right angles. We use them all the time and they are so important (if we are indoors, we see so many right angles), yet has there ever been a perfect right angle? What about a line, or a point? What are these things? We will discuss this in more detail in the *Chapter of Constructions*.

Of course, there are many other fun philosophical questions to discuss with students to get them thinking. For example, consider this classic question. Let's say you have a new ship. After some time, one of its boards needs replacing, so you do that. Is it still the same ship? Most would say that it is. What if we replace another board, and then another, and so on until one day the entire ship has been replaced? Is it still the same ship, or is it new? If it is new, at what point is it a different ship? Now we take this to the student. Every day, some atoms and molecules leave your body and are replaced by others. You have no atoms in your body from when you were little; they have slowly been replaced (98% of molecules in the human body are replaced each year!). So, are you still you? Are you a different person every time an atom is replaced? Maybe in a way you are different, but you are still *you*, right? If you are always you, but none of your parts are the same, then what are you? Where are you? This is a really fun discussion to have with students.

Getting them to think about truth and what is real is a great start an introduction to geometry. How can we find answers to these questions? We have to start looking at definitions again and start to draw conclusions based on basic facts. This is how mathematics (and indeed, geometry) has been built, starting with the Greeks. Making this connection with the practical world, asking philosophical questions, and then saying that we can try to come up with answers, not just guesses, gives context for why we would want to use deductive reasoning in geometry. Even if students are not fully ready for this approach, introducing these ideas to young adolescents is not only appropriate, but essential. Then, when they have the natural desire to study and learn in this way, the proper groundwork has been laid.

GUESS MY NUMBER GAME

This is the first and simplest of several games and puzzles presented in this chapter to help students develop their logical thinking skills. This game is an exercise in logical thinking in which students deduce numbers using guesses and information from structured responses.

Students of all ages enjoy playing this game as individuals, in small groups, or as a class. It can be done at any time, over a period of weeks or months or years. The game helps them to develop logical thinking in a way that will be important for when they start to write proofs. Essentially, a proof is an explanation of why something must be true. This game lays the foundations for the two main types of proofs students will do: direct proof and proof by contradiction. It is important that when students give their guesses, we ask them why they guessed what they did. I will explain the game and then give examples showing what questions to ask and what students may say.

PRESENTATION

We start on the board making a three-column chart. The first column is labeled "Guess," the second "Digits," and the last "Places." These can also be abbreviated as students become more familiar with the game.

Guess	Digits	Places

I tell the students that I am thinking of a two-digit number with no repeated digits (in this case, 68). The students make a guess, which is written in the Guess column. Perhaps the students guess 16. Since one of the digits in 16 matches one of the digits in 68, I will write a 1 in the Digits column. However, since the digit that has been guessed (6) is not in the correct place (it was guessed to be in the units place whereas it should be in the tens place), a 0 is written in the Places column.

Guess	Digits	Places
16	1	0

The game continues in this manner until the students have guessed the correct number.

Variations in order of increasing difficulty:

- Two-digit numbers, no repeats
- Two-digit numbers with possible repeats
- Three-digit numbers, no repeats
- Three-digit numbers with possible repeats
- Four-digit numbers

For a small group of students, they can all work together to come up with the best guesses. As the moderator, I will often ask a person to be the spokesperson for the group so that the shouting of guesses does not become unwieldy. As I listen to students' conversations, I can ask the individual students questions based on what I hear. When playing as a class, the goal is to get the number in as few guesses as possible. If the class is larger, or for variety, the students can work in teams, alternating guesses between them, where the team that identifies the mystery number is the winner. Notably, as students become more sophisticated, they know that a good guess will reveal information to their opponent, and sometimes it will reveal winning information, so they do not always optimally guess! This is still great logical and deductive reasoning.

EXAMPLE

MYSTERY NUMBER 83

Teacher "What would you like your first guess to be?"

Students "12!"

Teacher (fill out chart G:12 D:0 P:0) "There is nothing correct. Was that a bad guess then?"

Students "No, it was actually good because we now can eliminate two digits; 1 and 2.."

Teacher "Ok, what is your next guess?"

Students "34!"

Teacher (G: 34 D:1 P:0) "Next guess?"

Students "53!"

Teacher "Why did you guess that?"

Students "We know one digit is right, but in the wrong spot. We are guessing the correct digit is 3 and are going to see if that's true."

Guess	Digits	Places
12	0	0
34	1	0
53	1	1

Teacher (G: 53 D:1 P:1) "So is 3 the correct digit in the right place?"

Students "Yes!"

Teacher "Are you sure? How do you know that the 5 is not in the correct place and that the 3 was the wrong digit from your guess of 34?"

Students "Because if the 5 is correct, then the 4 would also have to be correct. That means the 4 is the correct digit but in the wrong place from the second clue, meaning that it must be in the tens place. But the 5 can't be in the tens place and the 4 in the tens place, so the 3 must be right."

Teacher "Okay. What would you like your next guess to be?"

Students "67!"

Teacher "But you know the 3 is correct. Why would you guess 67?"

Students "The 3 is correct, but the tens digit could be 6, 7, 8, or 9, which is a lot of possibilities. We could get lucky and guess it, but we can test more digits faster this way."

Teacher "Why couldn't 0 also be a digit to test?"

Students "Because you said it was a two-digit number and two-digit numbers can't start with 0."

Teacher "What would D:1 P1: tell you?"

Students "That the number is 63."

Teacher "What would D:1 P0 tell you?"

Students "That the number is 73."

Teacher "Why?"

Students "The ones digit can't be the correct digit in the correct place because we know that has to be 3. So, if there is a digit that is correct but in the wrong place, it has to be the units digit, which is 7. The 7 has to go in the tens place."

Teacher (G: 67 D:0 P:0) "Now what?"

Students "83!"

Teacher "Why not 89?"

Students "There are only two possibilities left; 83 or 93. If we guess 89, we'll know for sure what the correct number is, but it will take two turns. By guessing 83, we might get lucky and get it right. If we guess wrong, we know the number has to be 93 and so it will be two guesses anyway."

Teacher (G: 83 D:2 P:2)

Students "Hooray!"

Guess	Digits	Places
12	0	0
34	1	0
53	1	1
67	0	0
83	2	2

NOTES

- This level of thinking will not always happen right away. Through discussion, students will start to ask themselves these questions and thus be on the path to thinking logically rather than guessing. Sometimes this involves talking through some of the desired dialogue to model how to formulate good questions with clear language, and to show them what is happening. Sometimes students try to explain things but have a difficult time ordering their thoughts, so it is good practice to summarize what we think students are saying aloud. This gives the student who formulated the idea a model of language for her to use, as well as helping the rest of the group understand that student's idea.
- We don't have to ask all these questions every time, but these are the types of things we could ask.

- It is a good idea to have the number picked out and written down ahead of time. However, if students are unusually good guessers, it is okay to change the number midstream if it makes the game more interesting and doesn't affect the other clues. I would only do this if students are still reliant on luck more than logic to guess the correct answers. If they feel their best strategy is to try to read the teacher's mind, we want to show them that there is a better way, though in the short term they might be lucky with their guesses.
- The two-digit game has limited options for determining the best guess. Most of that was outlined in the example game above. As students get the hang of it, we can use repeated digits, as it allows for more possibilities in the clues. Once students have the pattern down of what to guess, and they can articulate their reasons, we can move on to three-digit numbers.
- It is important in this process that students give their reasoning. They are giving a proof of what they know. Their answer to the question, "Why couldn't 0 be a digit to also test?" was an example of a direct proof. The answer to the question, "How do you know that the 5 is not in the correct place and that the 3 was the wrong digit from your guess of 34?" was a proof by contradiction. This is normally a difficult concept, but it can occur naturally from the students in this context. When we do geometric proofs later and need a proof by contradiction, we can always recall this game to show how that type of proof works.
- When students guess, they look a few steps ahead to determine what might happen and what that would mean. This is another powerful tool that will be needed when doing formal proofs.
- Students can play the game without the teacher's help by having a student as the moderator.
- Especially when students are documenting their games, a great maxim is, "Write not to be understood, but so that it is impossible to misunderstand."
- Students will often assume things are true that aren't, and we need to question them when this happens so they see that they should conclude, not assume. If students aren't assuming, a good line of questioning would be to ask them why we don't know something is true.

EXTENSIONS

A way to target specific thinking, to help students work by themselves, and to help them move towards writing is to set up scenarios for them to puzzle out. Here are a few examples of questions for which they could provide written responses. When students write, that is a chance for us to work on refining that thought process and to encourage them to write in a clear and concise manner.

QUESTION 1 **Franklin was playing the number guess game with two-digit numbers and repeated numbers allowed.**

Guess	Digits	Places
35	0	0
14	1	0

What would be a better next guess, 17 or 71? Why?

Answer: If you guess 71 and get (1, 0) again, then you know for sure the number is 47 because if 1 was a correct digit, it would have been in the right place somewhere. Since it is not, the seven must be correct AND the 4 must also be a correct digit but in the wrong place. If you get (1, 1), that means the 1 is in the correct place. If you guess 17 and get (0, 0), then you know the first digit must be a 4. If you guess 17 and get (1, 0), then you know 1 must be the second digit. If you get (1, 1), then 7 must be in the second place, and if you get (0, 0), the first digit must be a 4. Since it is unlikely that 47 is the correct number, I would rather guess 17, as it more likely to give useful information.

QUESTION 2 **In a number guess game with three digits and no repeats, here are the guesses and results so far.**

Guess	Digits	Places
348	0	0
105	1	1
967	1	0

What digit MUST be in the number and why?

Answer: 2, since nine digits were guessed and only two have been correct, and there can be no repeats.

QUESTION 3 **Continuing the game above, the next guess and results are listed below. What must the three-digit number be and why?**

Guess	Digits	Places
348	0	0
105	1	1
967	1	0
702	2	0

Answer: The number must be 275. We know 2 has to be a correct digit from the first three guesses. Since 0 is in the same place in guess #2 and guess #4, but guess #4 has no digits in the correct place, 0 is not a digit. Since guess #4 has two digits correct, 7 must also be a correct digit. Since in guess #3 and guess #4, 7 is not in the correct place, it must be in the second place. From guess #2, the digit in the correct place must be 5, and therefore 2 must go in the first place. Therefore, 275 must be correct.

OTHER GAMES

Here are some other games for students to practice logical thinking. These can be played as a class or done individually with writing. It is fine for games to be played just for enjoyment sometimes, because we know that we can analyze later and trust that the logical thinking is happening. However, at some point we want to expand that into coherent oral and eventually written arguments, though it does not have to be done all the time.

COLOR SQUARE GAME

This game is similar to Guess My Number but with fewer possibilities. A square grid is made, usually 3×3 or 4×4. In a 3×3 game, three squares will be blue, three green, and three red (any colors can be used). However, all the squares of the same color must be adjacent. Two example solutions of a 3×3 grid are shown below.

G	G	B
R	G	B
R	R	B

R	B	B
R	R	B
G	G	G

The game starts with a blank grid. Students then request information about what is in a certain row or column. We then tell them what is in the column, but not the correct order (it is recommended to give the clues in alphabetical order to be consistent). The goal is for the students to deduce which squares are which colors in the fewest number of guesses. Here is what the information would look like for each of the example grids above.

	1G 2R	2G 1R	3B
1B 2G			
1B 1G 1R			
1B 2R			

	1G 2R	1B 1G 1R	2B 1G
2B 1R			
1B 2R			
3G			

With a 3×3, there are limited possibilities; there must always be a row of three of one color, and students can explain why this is after they have played the game for a while. Here is an example game.

Teacher "What would you like to know, a row or column?"

Students "The first row."

Teacher "What do we know?"

Students "That the middle square has to be green."

1B 2G			

Teacher "Why?"

Students "There are three green squares and they have to connect, so the blue couldn't be between them. We don't know if the blue is on the left or the right."

1B 2G		G	

Teacher "What would you like to know now?"

Students "The first column."

	1G 2R		
1B 2G		G	

Teacher "What do we know now?"

Students "That the top left square is green and the other two squares in that column are red. Again, the reds can't be separate and the green has to be next to the other green. Then we also know that the upper right square is blue because of the first clue."

	1G 2R		
1B 2G	G	G	B
	R		
	R		

Teacher "Do we know anything else for sure?"

Students "Yes! Now we know that the middle square has to be green because they all have to be attached."

	1G 2R		
1B 2G	G	G	B
	R	G	
	R		

Students "And wait! The bottom middle has to be red, which means the whole left column is blue!"

	1G 2R		
1B 2G	G	G	B
	R	G	B
	R	R	B

Games with a 3×3 grid can be done in two requests but it often takes three. Depending on what information is given by the first request, it is often better to ask for information of a particular row or column. The second request for row and column information may be random; however, information about a third row or column should always be based on previously gathered information. We should therefore be asking students why they requested certain information after their third request, and oftentimes after their second guess as well.

A 4×4 grid allows for much more variety of shapes of the colored squares, and thus more in-depth analysis of which row or column to guess. Here are a few examples of completed boards with yellow as the fourth color.

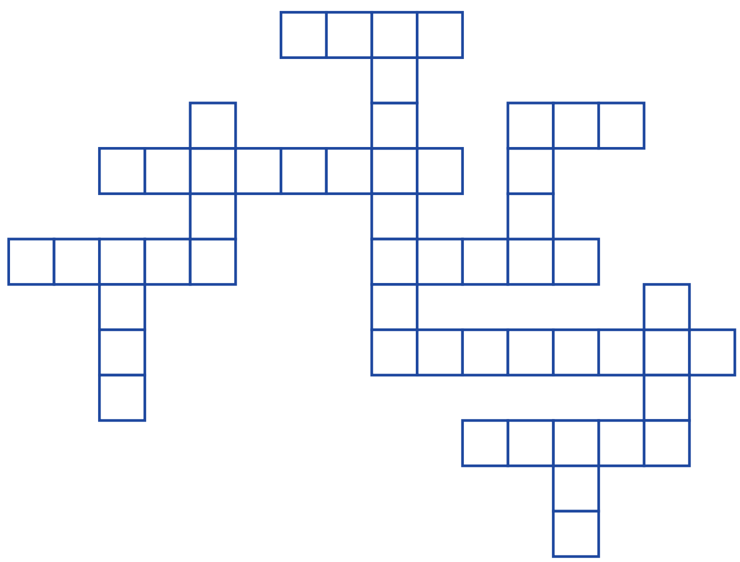
G	G	R	R
B	G	G	R
B	Y	Y	R
B	B	Y	Y

B	B	B	Y
B	G	Y	Y
G	G	R	Y
G	R	R	R

R	R	R	R
B	Y	Y	Y
B	Y	G	G
B	B	G	G

CLUELESS CROSSWORDS

This is an exercise where students have a crossword puzzle to fill in where only the words are given. Students must use logic to fill in the puzzle. For example, if there is only one seven-letter word, it must go in the only spot available. If there are two seven-letter words and the last letter of one of those must also be the second letter of a five-letter word, then we can look and see which it must be. Here is an example of a puzzle.



ARM	FREEDOMS	TELLS	TOUTS
BAT	GEOMETRY	TEST	USES
BELL	HUGE	TOAST	WETS
			YPERITES

Often figuring out where to start is a key to solving these efficiently. This is a good strategy for later when doing proofs. Usually there are several ways we can start, but finding some simple things to solve first is a good strategy. This is especially true when students have a complex diagram with limited angle measurements known, and they have to find the measure of a certain angle. It is often easier to start with a few simple known things and to work one's way towards the unknown quantity rather than try to immediately solve for it.

ADDITIONAL GAMES

Many other games and activities can help students in this type of thinking. Most of these can be done by students on their own or with classmates. It is good to have games and puzzles available to students at all times. Sometimes they need a break from the regular routine of class and so having these available is a great way for them to have a change of pace while doing something critical for developing their minds.

Sudoku

This game allows for logical thinking in a similar way to the Color Square game. Students can do these individually or in groups. Booklets at a variety of skill levels can be found readily in bookstores and online.

Logic Puzzles with Grids

These are great for deductive reasoning. They can vary in length and difficulty. Many of these have organizational structures to help students with their thought process. Here is a simple example from brainzilla.com/logic/logic-grid. More can be found on this website, or in puzzle books at a local bookstore.

This is a very easy logic grid puzzle. You may be able to solve this using less than one minute.

1. Peter's birthday is in April.
2. Eric is 7.
3. Arnold's birthday is in September.
4. Peter is 8.

		Ages			**Birthdays**		
		7 years	8 years	9 years	April	January	September
Names	Arnold						
	Eric						
	Peter						
Birthdays	April						
	January						
	September						

Names	**Ages**	**Birthdays**
Arnold		
Eric		
Peter		

Mastermind

This is the classic board game similar to the Color Square game, but it is done with colored pegs instead of numbers.

Guess Who?

This is a simpler game than Mastermind.

Number Slide Puzzle

This game is about trying to order the numbers by thinking ahead. These can be found in a variety of styles and sizes. The “Giant 27” puzzle below offers both possible and impossible challenges. For the impossible ones, it is great for students to try to figure out *why* they are impossible.

Othello (Reversi)

This is a classic game where the winning strategy may be counter-intuitive. The goal is to try to get the most chips to your color, but early in the game that is not as important as controlling certain regions of the board. Students have to think a few steps ahead and then work backwards to master controlling the board.

Inner Circle

This board game from the 1980s is a simple game where a backwards-thinking strategy is helpful to win.

Many other games involve strategy such as mancala, go, checkers, chess, the dots and boxes game, and so on.