# CONTINUOUS INTEREST

## PREREQUISITES

*Compound Interest, Discovering the Number e.*
If students have not discovered $e$, then this lesson could serve as an introduction to it.

## MATERIALS

Calculator
Charts

## PRESENTATION

Ask students to complete a chart such as the following:

*$1000 is invested in an account that yields 6% interest. Complete the chart below by finding the final amount after one year if it is compounded over the various timeframes.*

| Compounded | Calculation | Final Amount (full decimals) |
|---|---|---|
| Annually | $1000(1.06)$ | 1060 |
| Biannually | $1000(1.03)^2$ | 1060.9 |
| Quarterly | $1000\left(1+\frac{0.06}{4}\right)^4$ | 1061.36355063 |
| Monthly | $1000\left(1+\frac{0.06}{12}\right)^{12}$ | 1061.67781186 |
| Weekly | $1000\left(1+\frac{0.06}{52}\right)^{52}$ | 1061.79981955 |
| Daily | $1000\left(1+\frac{0.06}{365}\right)^{365}$ | 1061.83131068 |
| Hourly | $1000\left(1+\frac{0.06}{8760}\right)^{8760}$ | 1061.83632836 |
| Per Minute | $1000\left(1+\frac{0.06}{8760\cdot 60}\right)^{(8760\cdot 60)}$ | 1061.83654291 |
| Per Second | $1000\left(1+\frac{0.06}{8760\cdot 3600}\right)^{(8760\cdot 3600)}$ | 1061.83654648 |
| Per Hundredth of a Second | $1000\left(1+\frac{0.06}{8760\cdot 3600\cdot 100}\right)^{(8760\cdot 3600\cdot 100)}$ | 1061.83654654 |
| Per Millionth of a Second | $1000\left(1+\frac{0.06}{8760\cdot 3600\cdot 1{,}000{,}000}\right)^{(8760\cdot 360\cdot 1{,}000{,}000)}$ | 1061.83654655 |

**SAY** "What is happening to the amounts here? They are going up, but there is not much of a difference as we increase $n$. Even though the amount is getting larger as $n$ increases, we will not have an infinite amount of money. There is some highest number that this will go to, but what will it be? We can make some guesses based on our chart, but we would like to calculate this for sure. But what are we calculating? We want to calculate what would happen if we *continuously* compounded the interest. In other words, we let $n = \infty$. Of course, then we would have an infinite number of compoundings. We can't get to infinity, but we could still see what number it is getting closer to. Let's examine our formula for compound interest to get a clue."

Write $A = P\left(1+\frac{r}{n}\right)^{nt}$.

**SAY** "If we look carefully, we see that we have 1 plus a fraction with $n$ for a denominator, and then a multiple of $n$ to the exponent. In other words, we can sort of see $\left(1+\frac{1}{n}\right)^{n}$ hidden in there. Do you remember what this becomes as $n$ gets larger and larger, towards infinity? That's right, the number $e$! So, the number $e$, which is about 2.718, will be involved."

We can guide students through or let them work out the formula through the following work:

We are given that $A = P\left(1+\frac{r}{n}\right)^{nt}$. Let $m = \frac{n}{r}$.

Then we have $A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{m}\right)^{mrt} = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$

As $m$ increases without bound (gets as large as we want), we know that

$\left(1+\frac{1}{m}\right)^{m} \to e$, so we have that $P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \to P[e]^{rt}$

**SAY** "So $A = Pe^{rt}$ would be how we could calculate continuous compounding, which is faster than daily, hourly, by-the-minute, per second, per millisecond—basically infinite compounding. Again, the result will not be infinite as it reaches an asymptote at $e$, and we call this *continuous compounding*. This is the limit to what we could reach. For a given interest rate, no bank could beat this yield!"

## FOLLOW-UPS

- Students can calculate total amounts directly from the formula.
- They can write about and explain the parts of the formula.
- They can prove the formula on their own.
- They can solve for the amount or the principal.
- They could try to find the time or the rate, which would lead to the need for logarithms.

- They can create their own questions to solve.
- They can develop other formulas for non-banking situations that involve continuous compounding.

1. A sum of $15,000 is invested at an annual percentage rate of 8%. Find the balance after five years if it is compounded continuously.
2. The number of flies in an experimental population after $t$ hours is given by $F(t) = 10e^{0.02t}$, $t \geq 0$
   a. Find the initial number of flies in the population.
   b. How large is the population of flies after 48 hours?
   c. Sketch the graph of $F$. Use the following table for help.

| $t$ | 0 | 5 | 10 | 20 | 40 | 60 |
|---|---|---|---|---|---|---|
| $10e^{0.02t}$ | | | | | | |

**NOTES**

- The same principle applies as far as the initial value is concerned; that is, when $t = 0$, what will remain will be $P$, or the initial amount, $C$.
- There are many applications other than interest that involve continuous compounding and that could be explored as well.

# Logarithmic Functions

## A HISTORY OF LOGARITHMS

This lesson is designed to be an introduction to the historical development and necessity of logarithms. It could be expanded upon, embellished, or done in multiple parts. It is a critical lesson to the full treatment of this topic. The paper titled *The Fascinating History of Logarithms* by Dr. Bob Stein[1] is the source for this lesson, and the first part of the paper is included in Appendix D. He wrote this paper to take some of the mystery out of a subject that was traditionally arid, abstract, and confusing. He believes that a historical context has many added benefits for the student. The lesson as written here can be added to or altered, as the level of detail is left up to the teacher and the interest of the students.

### PREREQUISITES

Knowledge of exponents, including *Fractional Exponents* (Ch. 6, Vol. 1)

**MATERIALS**

Chuquet chart
Napier's bones
Slide rule

1. Stein, Robert. *The Fascinating History of Logarithms*.